Exercise 3E

1 **a** $z^4 - 1 = 0$

$z^4 = 1$

$z^4 = \cos 0 + \mathrm{i}\sin 0$

$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^4 = \cos\left(0 + 2k\pi\right) + \mathrm{i}\sin\left(0 + 2k\pi\right),\ k \in \mathbb{Z}$

Using De Moivre's theorem gives: $r^4\left(\cos 4\theta + \mathrm{i}\sin 4\theta\right) = \cos\left(0 + 2k\pi\right) + \mathrm{i}\sin\left(0 + 2k\pi\right),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r = 1$

Comparing the argument on both sides gives:

$2k\pi = 4\theta \Rightarrow k\pi = 2\theta$

When $k = 0$:

$0 = 2\theta \Rightarrow \theta = 0$ so $z_0 = \cos 0 + \mathrm{i}\sin 0 = 1$

When $k = 1$:

$\pi = 2\theta \Rightarrow \theta = \frac{\pi}{2}$ so $z_1 = \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right) = \mathrm{i}$

When $k = 2$:

$2\pi = 2\theta \Rightarrow \theta = \pi$ so $z_2 = \cos\pi + \mathrm{i}\sin\pi = -1$

When $k = 3$:

$3\pi = 2\theta \Rightarrow \theta = \frac{3\pi}{2}$ so $z_3 = \cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right) = -\mathrm{i}$

So $z = 1$, $z = -1$, $z = \mathrm{i}$ or $z = -\mathrm{i}$

1 b $z^3 - \mathrm{i} = 0$

$z^3 = \mathrm{i}$

Modulus = 1

Argument $= \frac{\pi}{2}$

So $z^3 = \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)$

$$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}$$

$$r^3\left(\cos 3\theta + \mathrm{i}\sin 3\theta\right) = \cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}$$

Comparing the modulus on both sides gives:

$r = 1$

Comparing the argument on both sides gives:

$$\frac{\pi}{2} + 2k\pi = 3\theta \Rightarrow \pi + 4k\pi = 6\theta \Rightarrow \theta = \frac{\pi + 4k\pi}{6}$$

When $k = 0$:

$$\theta = \frac{\pi}{6} \text{ so } z_0 = \cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$$

When $k = 1$:

$$\theta = \frac{5\pi}{6} \text{ so } z_1 = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$$

When $k = 2$:

$$\theta = \frac{3\pi}{2} \text{ so } z_2 = \cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right) = -\mathrm{i}$$

So $z = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$, $z = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$ or $z = -\mathrm{i}$

1 c $z^3 = 27$

$z^3 = 27(\cos 0 + \mathrm{i}\sin 0)$

$(r(\cos\theta + \mathrm{i}\sin\theta))^3 = 27(\cos(0+2k\pi) + \mathrm{i}\sin(0+2k\pi)),\ k \in \mathbb{Z}$

Using De Moivre's theorem gives: $r^3(\cos 3\theta + \mathrm{i}\sin 3\theta) = 27(\cos(0+2k\pi) + \mathrm{i}\sin(0+2k\pi)),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r = 3$

Comparing the argument on both sides gives:

$2k\pi = 3\theta \Rightarrow \theta = \frac{2k\pi}{3}$

When $k = 0$:

$\theta = 0$ so $z_0 = 3(\cos 0 + \mathrm{i}\sin 0) = 3$

When $k = 1$:

$\theta = \frac{2\pi}{3}$ so $z_1 = 3\left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$

When $k = 2$:

$\theta = \frac{4\pi}{3}$ so $z_2 = 3\left(\cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right) = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)$

So $z = 3,\ z = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$ or $z = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

1 d $z^4+64=0$

$z^4=-64$

Therefore:

$z^2=8\text{i}$ or $z^2=-8\text{i}$

When $z^2=8\text{i}$:

$$\left(r\left(\cos\theta+\text{i}\sin\theta\right)\right)^2=8\left(\cos\left(\frac{\pi}{2}+2k\pi\right)+\text{i}\sin\left(\frac{\pi}{2}+2k\pi\right)\right)$$

Using De Moivre's theorem gives: $r^2\left(\cos 2\theta+\text{i}\sin 2\theta\right)=8\left(\cos\left(\frac{\pi}{2}+2k\pi\right)+\text{i}\sin\left(\frac{\pi}{2}+2k\pi\right)\right)$

Comparing the modulus on both sides gives:

$r^2=8\Rightarrow r=2\sqrt{2}$

Comparing the argument on both sides gives:

$2\theta=\frac{\pi}{2}+2k\pi\Rightarrow\theta=\frac{\pi+4k\pi}{4}$

When $k=0$:

$\theta=\frac{\pi}{4}$ so $z_1=2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+\text{i}\sin\left(\frac{\pi}{4}\right)\right)=2+2\text{i}$

When $k=1$:

$\theta=\frac{5\pi}{4}$ so $z_2=2\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+\text{i}\sin\left(\frac{5\pi}{4}\right)\right)=-2-2\text{i}$

When $z^2=-8\text{i}$:

$$\left(r\left(\cos\theta+\text{i}\sin\theta\right)\right)^2=8\left(\cos\left(\frac{3\pi}{2}+2k\pi\right)+\text{i}\sin\left(\frac{3\pi}{2}+2k\pi\right)\right)$$

Using De Moivre's theorem gives: $r^2\left(\cos 2\theta+\text{i}\sin 2\theta\right)=8\left(\cos\left(\frac{3\pi}{2}+2k\pi\right)+\text{i}\sin\left(\frac{3\pi}{2}+2k\pi\right)\right)$

Comparing the modulus on both sides gives:

$r^2=8\Rightarrow r=2\sqrt{2}$

Comparing the argument on both sides gives:

$2\theta=\frac{3\pi}{2}+2k\pi\Rightarrow\theta=\frac{3\pi+4k\pi}{4}$

When $k=0$:

$\theta=\frac{3\pi}{4}$ so $z_3=2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+\text{i}\sin\left(\frac{3\pi}{4}\right)\right)=-2+2\text{i}$

When $k=1$:

$\theta=\frac{7\pi}{4}$ so $z_4=2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+\text{i}\sin\left(\frac{7\pi}{4}\right)\right)=2-2\text{i}$

So $z=2+2\text{i}$, $z=-2-2\text{i}$, $z=-2+2\text{i}$ or $z=2-2\text{i}$

1 e $z^4 + 4$

$z^4 = -4$

$z^2 = 2\text{i}$ or $z^2 = -2\text{i}$

When $z^2 = 2\text{i}$:

$$\left(r\left(\cos\theta + \text{i}\sin\theta\right)\right)^2 = 2\left(\cos\left(\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{2} + 2k\pi\right)\right)$$

Using De Moivre's theorem gives: $r^2\left(\cos 2\theta + \text{i}\sin 2\theta\right) = 2\left(\cos\left(\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{2} + 2k\pi\right)\right)$

Comparing the modulus on both sides gives:

$r^2 = 2 \Rightarrow r = \sqrt{2}$

Comparing the argument on both sides gives:

$2\theta = \frac{\pi}{2} + 2k\pi \Rightarrow \theta = \frac{\pi + 4k\pi}{4}$

When $k = 0$:

$\theta = \frac{\pi}{4}$ so $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \text{i}\sin\left(\frac{\pi}{4}\right)\right) = 1 + \text{i}$

When $k = 1$:

$\theta = \frac{5\pi}{4}$ so $z_2 = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + \text{i}\sin\left(\frac{5\pi}{4}\right)\right) = -1 - \text{i}$

When $z^2 = -2\text{i}$:

$$\left(r\left(\cos\theta + \text{i}\sin\theta\right)\right)^2 = 2\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right)$$

Using De Moivre's theorem gives: $r^2\left(\cos 2\theta + \text{i}\sin 2\theta\right) = 2\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right)$

Comparing the modulus on both sides gives:

$r^2 = 2 \Rightarrow r = \sqrt{2}$

Comparing the argument on both sides gives:

$2\theta = \frac{3\pi}{2} + 2k\pi \Rightarrow \theta = \frac{3\pi + 4k\pi}{4}$

When $k = 0$:

$\theta = \frac{3\pi}{4}$ so $z_1 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \text{i}\sin\left(\frac{3\pi}{4}\right)\right) = -1 + \text{i}$

When $k = 1$:

$\theta = \frac{7\pi}{4}$ so $z_2 = \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + \text{i}\sin\left(\frac{7\pi}{4}\right)\right) = 1 - \text{i}$

So $z = 1 + \text{i}$, $z = -1 - \text{i}$, $z = -1 + \text{i}$ or $z = 1 - \text{i}$

1 f $z^3 + 8\mathrm{i} = 0$

$z^3 = -8\mathrm{i}$

$(r(\cos\theta + \mathrm{i}\sin\theta))^3 = 8\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right)$

Using De Moivre's theorem gives: $r^3(\cos 3\theta + \mathrm{i}\sin 3\theta) = 8\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right)$

Comparing the modulus on both sides gives:

$r^3 = 8 \Rightarrow r = 2$

Comparing the argument on both sides gives:

$3\theta = \frac{3\pi}{2} + 2k\pi \Rightarrow \theta = \frac{3\pi + 4k\pi}{6}$

When $k = 0$:

$\theta = \frac{\pi}{2}$ so $z_1 = 2\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right) = 2\mathrm{i}$

When $k = 1$:

$\theta = \frac{7\pi}{6}$ so $z_2 = 2\left(\cos\left(\frac{7\pi}{6}\right) + \mathrm{i}\sin\left(\frac{7\pi}{6}\right)\right) = -\sqrt{3} - \mathrm{i}$

When $k = 2$:

$\theta = \frac{11\pi}{6}$ so $z_3 = 2\left(\cos\left(\frac{11\pi}{6}\right) + \mathrm{i}\sin\left(\frac{11\pi}{6}\right)\right) = \sqrt{3} - \mathrm{i}$

So $z = 2\mathrm{i}$, $z = \sqrt{3} - \mathrm{i}$, or $z = -\sqrt{3} - \mathrm{i}$

2 a $z^7 = 1$

Modulus = 1

Argument = 0

$\left(r\left(\cos\theta + i\sin\theta\right)\right)^7 = \cos(0+2k\pi) + i\sin(0+2k\pi),\ k \in \mathbb{Z}$

Using De Moivre's theorem gives: $r^7\left(\cos 7\theta + i\sin 7\theta\right) = \cos(0+2k\pi) + i\sin(0+2k\pi),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r = 1$

Comparing the argument on both sides gives:

$7\theta = 2k\pi \Rightarrow \theta = \dfrac{2k\pi}{7}$

When $k = 0$:

$\theta = 0$ so $z_0 = \cos 0 + \mathrm{i}\sin 0$

When $k = 1$:

$\theta = \dfrac{2\pi}{7}$ so $z_1 = \cos\left(\dfrac{2\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{2\pi}{7}\right)$

When $k = 2$:

$\theta = \dfrac{4\pi}{7}$ so $z_2 = \cos\left(\dfrac{4\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{4\pi}{7}\right)$

When $k = 3$:

$\theta = \dfrac{6\pi}{7}$ so $z_3 = \cos\left(\dfrac{6\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{6\pi}{7}\right)$

When $k = 4$:

$\theta = \dfrac{8\pi}{7}$ so $z_4 = \cos\left(\dfrac{8\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{8\pi}{7}\right) = \cos\left(-\dfrac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\dfrac{6\pi}{7}\right)$

When $k = 5$:

$\theta = \dfrac{10\pi}{7}$ so $z_5 = \cos\left(\dfrac{10\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{10\pi}{7}\right) = \cos\left(-\dfrac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\dfrac{4\pi}{7}\right)$

When $k = 6$:

$\theta = \dfrac{12\pi}{7}$ so $z_6 = \cos\left(\dfrac{12\pi}{7}\right) + \mathrm{i}\sin\left(\dfrac{12\pi}{7}\right) = \cos\left(-\dfrac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\dfrac{2\pi}{7}\right)$

2 **b** $z^4 + 16\text{i} = 0$

$z^4 = -16\text{i}$

Modulus = 16

Argument = $\frac{3\pi}{2}$

$$\left(r(\cos\theta + i\sin\theta)\right)^4 = 16\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right),\ k \in \mathbb{Z}$$

Using De Moivre's theorem gives

$$r^4(\cos 4\theta + \text{i}\sin 4\theta) = 16\left(\cos\left(\frac{3\pi}{2} + 2k\pi\right) + \text{i}\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right),\ k \in \mathbb{Z}$$

Comparing the modulus on both sides gives:

$r = 2$

Comparing the argument on both sides gives:

$$4\theta = \frac{3\pi}{2} + 2k\pi \Rightarrow \theta = \frac{3\pi + 4k\pi}{8}$$

When $k = 0$:

$$\theta = \frac{3\pi}{8} \text{ so } z_0 = 2\left(\cos\left(\frac{3\pi}{8}\right) + \text{i}\sin\left(\frac{3\pi}{8}\right)\right)$$

When $k = 1$:

$$\theta = \frac{7\pi}{8} \text{ so } z_1 = 2\left(\cos\left(\frac{7\pi}{8}\right) + \text{i}\sin\left(\frac{7\pi}{8}\right)\right)$$

When $k = 2$:

$$\theta = \frac{11\pi}{8} \text{ so } z_2 = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$$

When $k = 3$:

$$\theta = \frac{15\pi}{8} \text{ so } z_3 = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$$

2 c $z^5 + 32 = 0$

$z^5 = -32$

Modulus = 32

Argument = π

$\left(r\left(\cos\theta + i\sin\theta\right)\right)^5 = 32\left(\cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right)\right),\ k \in \mathbb{Z}$

Using De Moivre's theorem gives: $r^5\left(\cos 5\theta + \mathrm{i}\sin 5\theta\right) = 32\left(\cos\left(\pi + 2k\pi\right) + \mathrm{i}\sin\left(\pi + 2k\pi\right)\right),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r^5 = 32 \Rightarrow r = 2$

Comparing the argument on both sides gives:

$5\theta = \pi + 2k\pi \Rightarrow \theta = \frac{\pi + 2k\pi}{5}$

When $k = 0$:

$\theta = \frac{\pi}{5}$ so $z_0 = 2\left(\cos\left(\frac{\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{5}\right)\right)$

When $k = 1$:

$\theta = \frac{3\pi}{5}$ so $z_1 = 2\left(\cos\left(\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(\frac{3\pi}{5}\right)\right)$

When $k = 2$:

$\theta = \pi$ so $z_2 = 2\left(\cos\pi + \mathrm{i}\sin\pi\right)$

When $k = -1$:

$\theta = -\frac{\pi}{5}$ so $z_{-1} = 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$

When $k = -2$:

$\theta = -\frac{3\pi}{5}$ so $z_0 = 2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

2 d $z^3 = 2 + 2\mathrm{i}$

Modulus = $\sqrt{2^2 + 2^2} = 2\sqrt{2}$

argument = $\frac{\pi}{4}$

$\left(r\left(\cos\theta + i\sin\theta\right)\right)^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right),\ k \in \mathbb{Z}$

Using De Moivre's theorem gives:

$r^3\left(\cos 3\theta + \mathrm{i}\sin 3\theta\right) = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + 2k\pi\right)\right),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r^3 = 2\sqrt{2} \Rightarrow r = \sqrt{2}$

Comparing the argument on both sides gives:

$3\theta = \frac{\pi}{4} + 2k\pi \Rightarrow \theta = \frac{\pi + 8k\pi}{12}$

When $k = 0$:

$\theta = \frac{\pi}{12}$ so $z_0 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$

When $k = 1$:

$\theta = \frac{3\pi}{4}$ so $z_1 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$

When $k = -1$:

$\theta = -\frac{7\pi}{12}$ so $z_1 = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$

2 e $z^4 + 2\mathrm{i}\sqrt{3} = 2$

$z^4 = 2 - 2\mathrm{i}\sqrt{3}$

Modulus = $\sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4$

argument $= \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right),\ k \in \mathbb{Z}$$

Using De Moivre's theorem gives:

$$r^4\left(\cos 4\theta + \mathrm{i}\sin 4\theta\right) = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right),\ k \in \mathbb{Z}$$

Comparing the modulus on both sides gives:

$r^4 = 4 \Rightarrow r = \sqrt{2}$

Comparing the argument on both sides gives:

$$4\theta = -\frac{\pi}{3} + 2k\pi \Rightarrow \theta = \frac{-\pi + 6k\pi}{12}$$

When $k = 0$:

$$\theta = -\frac{\pi}{12} \text{ so } z_0 = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$$

When $k = 1$:

$$\theta = \frac{5\pi}{12} \text{ so } z_1 = \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right)$$

When $k = 2$:

$$\theta = \frac{11\pi}{12} \text{ so } z_2 = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right)$$

When $k = -1$:

$$\theta = -\frac{7\pi}{12} \text{ so } z_1 = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

2 f $z^3 + 32\sqrt{3} + 32\mathrm{i} = 0$

$z^3 = -32\sqrt{3} - 32i$

Modulus = $= \sqrt{\left(-32\sqrt{3}\right)^2 + (-32)^2} = 64$

Argument = $-\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\frac{5\pi}{6}$

$\left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right),\ k \in \mathbb{Z}(\ )$

Using De Moivre's theorem gives:

$r^3(\cos 3\theta + \mathrm{i}\sin 3\theta) = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right),\ k \in \mathbb{Z}$

Comparing the modulus on both sides gives:

$r^3 = 64 \Rightarrow r = 4$

Comparing the argument on both sides gives:

$3\theta = -\frac{5\pi}{6} + 2k\pi \Rightarrow \theta = \frac{-5\pi + 12k\pi}{18}$

When $k = 0$:

$\theta = -\frac{2\pi}{9}$ so $z_0 = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

When $k = 1$:

$\theta = \frac{4\pi}{9}$ so $z_1 = 4\left(\cos\left(\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(\frac{7\pi}{18}\right)\right)$

When $k = -1$:

$\theta = -\frac{8\pi}{9}$ so $z_{-1} = 4\left(\cos\left(-\frac{17\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

3 a $z^4 = 3 + 4\mathrm{i}$

Modulus = $= \sqrt{3^2 + 4^2} = 5$

Argument = $\tan^{-1}\left(\frac{4}{3}\right) = 0.927...$

$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^4 = 5\mathrm{e}^{(0.927...+2k\pi)}$

$r^4\mathrm{e}^{4\mathrm{i}\theta} = 5\mathrm{e}^{(0.927...+2k\pi)}$

$r^4 = 5 \Rightarrow r = 5^{0.25}$

$4\theta = 0.927... + 2k\pi$

When $k = 0$:

$\theta = 0.231...$, so $z_1 = 5^{0.25}\mathrm{e}^{0.23\mathrm{i}}$

When $k = 1$:

$\theta = 1.802...$, so $z_2 = 5^{0.25}\mathrm{e}^{1.80\mathrm{i}}$

When $k = -1$:

$\theta = -1.338...$, so $z_3 = 5^{0.25}\mathrm{e}^{-1.34\mathrm{i}}$

When $k = -2$:

$\theta = -2.909...$, so $z_4 = 5^{0.25}\mathrm{e}^{-2.91\mathrm{i}}$

3 b $z^3 = \sqrt{11} - 4\text{i}$

Modulus $= \sqrt{\left(\sqrt{11}\right)^2 + (-4)^2} = 3\sqrt{3}$

Argument $= \tan^{-1}\left(-\frac{4}{\sqrt{11}}\right) = -0.878...$

$\left(re^{i\theta}\right)^3 = 3\sqrt{3}e^{(-0.878...+2k\pi)}$

$r^3 e^{3i\theta} = 3\sqrt{3}e^{(-0.878...+2k\pi)}$

$r^3 = 3\sqrt{3} \Rightarrow r = \sqrt{3}$

$3\theta = -0.878... + 2k\pi$

When $k = 0$:

$\theta = -0.292...$, so $z_1 = \sqrt{3}e^{-0.29i}$

When $k = 1$:

$\theta = 1.801...$, so $z_2 = \sqrt{3}e^{1.80i}$

When $k = -1$:

$\theta = -2.387...$, so $z_3 = \sqrt{3}e^{-2.39i}$

c $z^4 = -\sqrt{7} + 3\text{i}$

Modulus $= = \sqrt{\left(-\sqrt{7}\right)^2 + 3^2} = 4$

Argument $= \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293...$

$\left(re^{i\theta}\right)^4 = 4e^{(2.293...+2k\pi)}$

$r^4 e^{4i\theta} = 4e^{(2.293...+2k\pi)}$

$r^4 = 4 \Rightarrow r = \sqrt{2}$

$4\theta = 2.293... + 2k\pi$

When $k = 0$:

$\theta = 0.573...$, so $z_1 = \sqrt{2}e^{0.57i}$

When $k = 1$:

$\theta = 2.144...$, so $z_2 = \sqrt{2}e^{2.14i}$

When $k = -1$:

$\theta = -0.997...$, so $z_3 = \sqrt{2}e^{-1.00i}$

When $k = -2$:

$\theta = -2.568...$, so $z_4 = \sqrt{2}e^{-2.57i}$

4 a $(z+1)^3 = -1$

$(z+1)^3 = (-1)^3$

$(z+1)^3 - (-1)^3 = 0$

$\left((z+1)-(-1)\right)\left((z+1)^2 + (-1)(z+1) + (-1)^2\right) = 0$

$(z+2)\left((z^2+2z+1) + (-z-1) + 1\right) = 0$

$(z+2)(z^2+z+1) = 0$

$(z+2) = 0$ has the solution $z = -2$

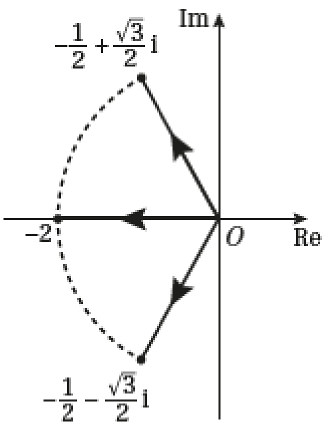
$z^2 + z + 1 = 0$ has solutions;

$z = \dfrac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{-3}}{2}$

$= \dfrac{-1 \pm i\sqrt{3}}{2}$

b



4 c The points $(-2,0)$, $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ lie on a circle.

Substituting these values into the general form of a circle $(x-a)^2+(y-b)^2=r^2$ gives:

For $(-2,0)$:

$(-2-a)^2+(0-b)^2=r^2$

$4+4a+a^2+b^2=r^2$

$a^2+b^2+4a+4=r^2$ **(1)**

For $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$:

$\left(-\frac{1}{2}-a\right)^2+\left(\frac{\sqrt{3}}{2}-b\right)^2=r^2$

$\frac{1}{4}+a+a^2+\frac{3}{4}-\sqrt{3}b+b^2=r^2$

$a^2+b^2+a-\sqrt{3}b+1=r^2$ **(2)**

For $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$:

$\left(-\frac{1}{2}-a\right)^2+\left(-\frac{\sqrt{3}}{2}-b\right)^2=r^2$

$\frac{1}{4}+a+a^2+\frac{3}{4}+\sqrt{3}b+b^2=r^2$

$a^2+b^2+a+\sqrt{3}b+1=r^2$ **(3)**

Subtracting **(3)** from **(2)** gives:

$-2\sqrt{3}b=0 \Rightarrow b=0$

Substituting $b=0$ into **(1)** gives:

$a^2+4a+4=r^2$ **(4)**

Substituting $b=0$ into **(3)** gives:

$a^2+a+1=r^2$ **(5)**

Subtracting **(5)** from **(4)** gives:

$3a+3=0 \Rightarrow a=-1$

Substituting $a=-1$ into **(5)** gives:

$r^2=1 \Rightarrow r=1$

Therefore the circle has centre $(-1,0)$ and radius 1

5 a $z^5 - 1 = 0$

$z^5 = 1$

Modulus = 1

Argument = 0

$(r(\cos\theta + \mathrm{i}\sin\theta))^5 = (\cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi)),\ k \in \mathbb{Z}$

$r^5(\cos 5\theta + \mathrm{i}\sin 5\theta) = (\cos(2k\pi) + \mathrm{i}\sin(2k\pi)),\ k \in \mathbb{Z}$

$r^5 = 1 \Rightarrow r = 1$

$5\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{5}$

When $k = 0$:

$z_1 = 1$

When $k = 1$:

$z_2 = \cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right)$

When $k = 2$:

$z_3 = \cos\left(\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(\frac{4\pi}{5}\right)$

When $k = -1$:

$z_4 = \cos\left(-\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{5}\right)$

When $k = -2$:

$z_5 = \cos\left(-\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{5}\right)$

b $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$1 + \cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(\frac{4\pi}{5}\right) + \cos\left(-\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{5}\right) = 0$

$1 + \cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - \mathrm{i}\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - \mathrm{i}\sin\left(\frac{4\pi}{5}\right) = 0$

$1 + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = 0$

$1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = 0$

$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$ as required.

6 a $-2 - 2\mathrm{i}\sqrt{3}$

Modulus $= \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$

Argument $= -\pi + \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{2\pi}{3}$

6 b $z^4 + 2 + 2\text{i}\sqrt{3} = 0$

$z^4 = -2 - 2\text{i}\sqrt{3}$

$\left(re^{\text{i}\theta}\right)^4 = 4e^{\left(-\frac{2\pi}{3}+2k\pi\right)}$

$r^4 e^{4\text{i}\theta} = 4e^{\left(-\frac{2\pi}{3}+2k\pi\right)}$

$r^4 = 4 \Rightarrow r = \sqrt{2}$

$4\theta = -\frac{2\pi}{3} + 2k\pi$

When $k = 0$:

$\theta = -\frac{\pi}{6}$, so $z_1 = \sqrt{2}e^{-\frac{\pi}{6}\text{i}}$

When $k = 1$:

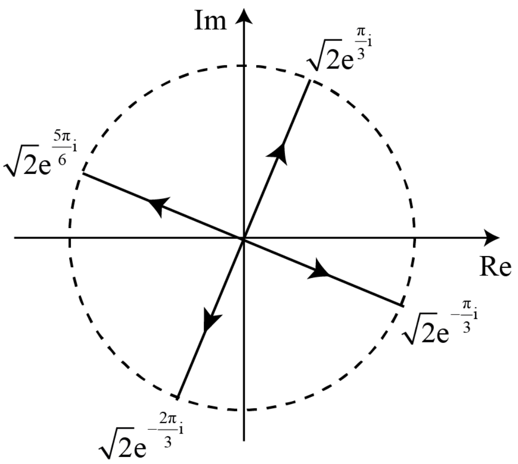
$\theta = \frac{\pi}{3}$, so $z_2 = \sqrt{2}e^{\frac{\pi}{3}\text{i}}$

When $k = 2$:

$\theta = \frac{5\pi}{6}$, so $z_3 = \sqrt{2}e^{\frac{5\pi}{6}\text{i}}$

When $k = -1$:

$\theta = -\frac{2\pi}{3}$, so $z_4 = \sqrt{2}e^{-\frac{2\pi}{3}\text{i}}$



7 $z^4 = 2\left(1-\mathrm{i}\sqrt{3}\right)$

Modulus $= \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4$

Argument $= \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^4 = 4\mathrm{e}^{\left(-\frac{\pi}{3}+2k\pi\right)}$

$r^4\mathrm{e}^{4\mathrm{i}\theta} = 4\mathrm{e}^{\left(-\frac{\pi}{3}+2k\pi\right)}$

$r^4 = 4 \Rightarrow r = \sqrt{2}$

$4\theta = -\frac{\pi}{3} + 2k\pi$

When $k = 0$:

$\theta = -\frac{\pi}{12}$, so $z_1 = \sqrt{2}\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}}$

When $k = 1$:

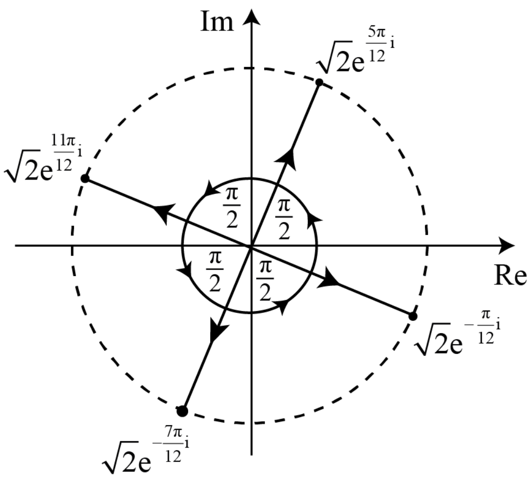
$\theta = \frac{5\pi}{12}$, so $z_2 = \sqrt{2}\mathrm{e}^{\frac{5\pi}{12}\mathrm{i}}$

When $k = 2$:

$\theta = \frac{11\pi}{12}$, so $z_3 = \sqrt{2}\mathrm{e}^{\frac{11\pi}{12}\mathrm{i}}$

When $k = -1$:

$\theta = -\frac{7\pi}{12}$, so $z_4 = \sqrt{2}\mathrm{e}^{-\frac{7\pi}{12}\mathrm{i}}$



8 a $z = \sqrt{6} + i\sqrt{2}$

Modulus $= \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} = 2\sqrt{2}$

Argument $= \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \frac{\pi}{6}$

b $z^4 = \left(\sqrt{6} + i\sqrt{2}\right)^4$

$= -32 + 32\sqrt{3}i$

Since $w^3 = z^4$

$w^3 = -32 + 32\sqrt{3}i$

Modulus $= \sqrt{(-32)^2 + \left(32\sqrt{3}\right)^2} = 64$

Argument $= \pi - \tan^{-1}\left(\frac{32\sqrt{3}}{32}\right) = \frac{2\pi}{3}$

$\left(re^{i\theta}\right)^3 = 64e^{\left(\frac{2\pi}{3} + 2k\pi\right)}$

$r^3 e^{3i\theta} = 64e^{\left(\frac{2\pi}{3} + 2k\pi\right)}$

$r^3 = 64 \Rightarrow r = 4$

When $k = 0$:

$\theta = \frac{2\pi}{9}$, so $w_1 = 4e^{\frac{2\pi}{9}i}$

When $k = 1$:

$\theta = \frac{8\pi}{9}$, so $w_2 = 4e^{\frac{8\pi}{9}i}$

When $k = -1$:

$\theta = -\frac{4\pi}{9}$, so $w_2 = 4e^{-\frac{4\pi}{9}i}$

9 **a** $1+z+z^2+z^3+z^4+z^5+z^6+z^7=0$

$\frac{(1-z^8)}{1-z}=0$

$z^8-1=0 \Rightarrow z^8=1$

$z^8=1(\cos 0+\mathrm{i}\sin 0)$

$(r(\cos\theta+\mathrm{i}\sin\theta))^8=1(\cos(0+2k\pi)+\mathrm{i}\sin(0+2k\pi)),\ k\in\mathbb{Z}$

Using De Moivre's theorem gives:

$r^8(\cos 8\theta+\mathrm{i}\sin 8\theta)=1(\cos(0+2k\pi)+\mathrm{i}\sin(0+2k\pi)),\ k\in\mathbb{Z}$

Comparing modulus:

$r^8=1 \Rightarrow r=1$

Comparing argument:

$8\theta=2k\pi \Rightarrow \theta=\frac{k\pi}{4}$

When $k=0$: $\theta=0 \Rightarrow z_1=\cos 0+\mathrm{i}\sin 0=1 \Rightarrow z_1=1$

When $k=1$: $\theta=\frac{\pi}{4} \Rightarrow z_2=\cos\left(\frac{\pi}{4}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}\right) \Rightarrow z_2=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$

When $k=2$: $\theta=\frac{\pi}{2} \Rightarrow z_3=\cos\left(\frac{\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\pi}{2}\right) \Rightarrow z_3=\mathrm{i}$

When $k=3$: $\theta=\frac{3\pi}{4} \Rightarrow z_4=\cos\left(\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(\frac{3\pi}{4}\right) \Rightarrow z_4=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$

When $k=4$: $\theta=\pi \Rightarrow z_5=\cos\pi+\mathrm{i}\sin\pi \Rightarrow z_5=-1$

When $k=5$: $\theta=\frac{5\pi}{4} \Rightarrow z_6=\cos\left(\frac{5\pi}{4}\right)+\mathrm{i}\sin\left(\frac{5\pi}{4}\right) \Rightarrow z_6=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}$

When $k=6$: $\theta=\frac{3\pi}{2} \Rightarrow z_7=\cos\left(\frac{3\pi}{2}\right)+\mathrm{i}\sin\left(\frac{3\pi}{2}\right) \Rightarrow z_7=-\mathrm{i}$

When $k=7$: $\theta=\frac{7\pi}{4} \Rightarrow z_8=\cos\left(\frac{7\pi}{4}\right)+\mathrm{i}\sin\left(\frac{7\pi}{4}\right) \Rightarrow z_8=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}$

b Expressing as a product of the factors:

$(z+1)(z-1)(z-\mathrm{i})(z+\mathrm{i})\left(z-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}\right)\left(z-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}\right)\left(z+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}\right)\left(z+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}\right)$

$=(z^2-1)(z^2+1)(z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1)$

$=(z^2-1)(z^2+1)(z^4+1)$

So (z^2+1) and (z^4+1) are factors

Challenge

a $z^6 = 1$
Modulus = 1
Argument = 0
$\left(re^{i\theta}\right)^6 = e^{(0+2k\pi)}$
$r^6 e^{6i\theta} = e^{2k\pi}$
$r^6 = 1 \Rightarrow r = 1$
$6\theta = 2k\pi \Rightarrow \theta = \frac{k\pi}{3}$
When $k = 0$:
$\theta = 0$, so $z_1 = 1$
When $k = 1$:
$\theta = \frac{\pi}{3}$, so $z_2 = e^{\frac{\pi}{3}i}$
When $k = 2$:
$\theta = \frac{2\pi}{3}$, so $z_3 = e^{\frac{2\pi}{3}i}$
When $k = 3$:
$\theta = \pi$, so $z_4 = e^{\pi i}$
When $k = -1$:
$\theta = -\frac{\pi}{3}$, so $z_5 = e^{-\frac{\pi}{3}i}$
When $k = -2$:
$\theta = -\frac{2\pi}{3}$, so $z_5 = e^{-\frac{2\pi}{3}i}$

b $(z+1)^6 = z^6 \Rightarrow \left(\frac{z+1}{z}\right)^6 = 1$

From part a

$$\frac{z+1}{z} = e^{\frac{k\pi}{3}i} \Rightarrow \frac{z+1}{z} = \cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}$$

$$z+1 = z\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right)$$

$$z = \frac{1}{\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3} - 1}$$

$$= \frac{1}{\left(\cos\frac{k\pi}{3} - 1\right) + i\sin\frac{k\pi}{3}} \times \frac{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}$$

$$= \frac{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}{\left(\cos\frac{k\pi}{3} - 1\right)^2 + \sin^2\frac{k\pi}{3}}$$

$$= \frac{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}{\cos^2\frac{k\pi}{3} - 2\cos\frac{k\pi}{3} + 1 + \sin^2\frac{k\pi}{3}}$$

$$= \frac{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}}$$

$$= \frac{\left(\cos\frac{k\pi}{3} - 1\right) - i\sin\frac{k\pi}{3}}{-2\left(\cos\frac{k\pi}{3} - 1\right)}$$

$$= -\frac{1}{2} + \frac{1}{2}\left(\frac{\sin\frac{k\pi}{3}}{\cos\frac{k\pi}{3} - 1}\right)$$

$$= -\frac{1}{2} + \frac{1}{2}\left(\frac{2\sin\frac{k\pi}{6}\cos\frac{k\pi}{6}}{1 - 2\sin^2\frac{k\pi}{6} - 1}\right)$$

$$= -\frac{1}{2} + \frac{1}{2}\left(\frac{2\sin\frac{k\pi}{6}\cos\frac{k\pi}{6}}{-2\sin^2\frac{k\pi}{6}}\right)$$

$$= -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{6}, k = 0, 1, 2, 3, 4, 5$$